§1.2 对等

Def. 若存在 $\varphi: A \to B$ 双射，则称 A 与 B 对等，记为 $A \sim B$.

例1 $[a,b] \overset{\varphi}{\sim} [0,1]$ $(0,1) \overset{\varphi}{\sim} (0,+\infty)$

例2 若 $A \overset{\varphi}{\sim} B$, $C \overset{\psi}{\sim} D$, $A \cap C = \phi$, $B \cap D = \phi$，则 $A \cup C \sim B \cup D$

Sol. $\exists f: A \cup C \to B \cup D$

$$f(x) = \begin{cases} \varphi(x), & x \in A \\ \psi(x), & x \in C \end{cases}$$

例3 若 $A \overset{\varphi}{\sim} B$, $C \overset{\varphi}{\sim} D$, $C \subseteq A$, $D \subseteq B$，则 $A \setminus C \sim B \setminus D$

Sol. $\forall x \in A \setminus C$，令 $f(x) = \varphi(x) \in B$，下证 $f(x) \in B \setminus D$：

由反证可知 $f(x) \in B \setminus D$

(若 $f(x) \in D$，则 $\exists !\, x_0 \in C$ s.t. $f(x_0) = \varphi(x) = \varphi(x_0)$)

于是 $A \setminus C \sim B \setminus D$

Rem. 若 $A \overset{\varphi}{\sim} B$, $C \overset{\psi}{\sim} D$, $C \subseteq A$, $D \subseteq B$，则 $A \setminus C \not\sim B \setminus D$

例如：$A = \{1,2,3,\cdots\}$, $B = \{5,6,7,\cdots\}$

$C = \{3,4,5,\cdots\}$, $D = \{8,9,10,\cdots\}$

有 $\varphi: A \to B$, $x \mapsto x+4$

$\psi: C \to D$, $x \mapsto x+5$

Prop. $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$

$\varphi(A \cap B) \subseteq \varphi(A) \cap \varphi(B)$

$\varphi^{-1}(A \cup B) = \varphi^{-1}(A) \cup \varphi^{-1}(B)$

$\varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B)$

Thm. 若 $A \sim B_0 \subseteq B$，则 $\overline{\overline{A}} \leq \overline{\overline{B}}$

Thm. Bernstein

若 $A \overset{f}{\sim} B_0 \subseteq B$，$B \overset{g}{\sim} A_0 \subseteq A$，则 $A \sim B$